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ON SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

by

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and

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I. Introduction

In the consideration of propagation of electromagnetic waves in inhomogeneous anisotropic stratified media, Maxwell's equations reduce to a system of four first order linear ordinary differential equations. The treatment of these equations has resulted in the present extension of the Peano-Baker method. However, since higher order systems of this type occur frequently in all branches of applied mathematics the theory is developed for a system of "n" equations. The application of the analysis to propagation in the ionosphere will appear in a subsequent report.

Consider the system of ordinary differential equations.

$$\frac{dU(z)}{dz} = A(z)\,U(z) \tag{1}$$

in which U(z) is an n-rowed column vector and A(z) is a square matrix function of z. When the matrix A is constant, the solutions of (1) are linear combinations of exponential functions with constant coefficients (if the latent roots of A are distinct) which are interpreted as damped, growing or constant oscillations or waves, or else represent non-oscillatory growth or decay. These solutions are so useful in applications that it is natural to seek similar solutions when A is not constant. Many attempts to obtain such solutions have been made and much of this work has often been unnecessarily repeated partly because there is no general treatment of the system (1) from this point of view.

The nearest approach to such a treatment is contained in the WKB method which yields an approximate solution of a linear second order equation as a sum of exponentials with variable coefficients and exponents. In some presentations of the method higher order approximations are also obtained, but it is generally not clear whether the successive approximations converge to a solution of the equation or not. For a large class of second order equations involving a parameter in a particular way, Birkhoff[8] and Langer[9] have settled this matter by showing that the WKB approximation is the leading term in an asymptotic expansion of the solution with respect to the parameter.

In all other cases, however, when the equation or system does not involve a parameter (or does but not in the appropriate manner) this result isn't applicable and the significance of the WKB approximation remains in doubt. But recently several workers have shown that the WKB approximation for the reduced one dimensional wave equation can be obtained as the first term in the application of the Picard iteration method to an equivalent system of integral equations[10]. Thus when the Picard method yields a solution, the WKB approximation is the first term in a convergent series. The successive terms in the series have been interpreted in terms of multiple reflections.

For systems or equations of order greater than two, there is no general analogue of the WKB method, although for equations containing a parameter in the appropriate manner Birkhoff's theorem assures the existence of an asymptotic expansion with respect to the parameter. The leading terms in the expansion are exponentials of the desired type, and thus provide a generalization of the WKB method to such equations. But for all those cases in which the system does not involve a parameter, there is no generalization available.

In this article we present what seems to be a new approach to the original problem of obtaining exponential-type solutions of the system (1). We begin with the Peano-Baker matrizant representation of the solution of (1), which is known to be uniformly convergent in a specified large region, although it converges very slowly and therefore doesn't exhibit the behavior of the solution. By applying certain transformations to this representation we obtain a representation of the desired type—the solution is a sum of exponentials with variable coefficients. This representation, which is of course still uniformly convergent in a large region, is rapidly convergent in many cases and furthermore exhibits clearly the behavior of the solution. Thus one of the weaknesses of the matrizant method, its slow convergence, is removed.

When the result is specialized to a second order system equivalent to a single second order equation, the first term in the representation is identical with the WKB approximation. Thus this representation may be considered to be the natural generalization of the WKB method to higher order systems. In addition it shows that the WKB approximation is the first term of a uniformly convergent representation of the solution, and the region of convergence is known. The representation fails, and must be modified, at points where the elementary divisors of A(z) are not simple i.e. at some points where two or more latent roots (eigenvalues) of A(z) are equal. In the second order case these are the ordinary turning-points, and thus we may also call them turning-points in the higher order case. The original matrizant representation is valid in a neighborhood of each such point, and thus provides the connection between the other representations of the rigorous solution in its vicinity.

If the system (1) involves a parameter in an appropriate way, then we can deduce an asymptotic expansion of the solution with respect to this parameter, from our representation. Thus we can essentially deduce Birkhoff's theorem concerning the existence of an asymptotic expansion and further obtain explicit expressions for the successive terms. In particular the first term is still the (generalized) WKB approximation, and thus we prove that this approximation is also the leading term in an asymptotic expansion of the solution provided that the system contains a parameter appropriately. The connection formulae for the asymptotic expansion are not considered.

The exponential terms can be readily interpreted as oscillations or waves, just as in the case where A is constant. Each exponent is the integral of one latent root of A and the coefficient of each exponential is variable, in fact it is a series of multiple integrals. These integrals have an immediate interpretation in terms of continuous multiple reflection caused by the variation of A(z). Thus it is generally not possible for a single wave to occur alone, i.e. to yield a solution itself. Instead any one wave gives rise to the others by the continuous reflections just mentioned. In addition "lumped" reflections may occur from regions between the turning points where the separate waves may be strongly "coupled" together. The overall effect of these reflections is described by the scattering matrix, which is defined and expressed in terms of our representation of the solution.

II The Peano-Baker Method

We wish to examine the solution of the system 1) for real and complex values of z.

1) $$\frac{dU(z)}{dz} = A(z)\,U(z) \quad ; \quad U(z_0) = U_0$$

In the complex case we require that:

(a) The n^2 elements $a_{ij}(z)$ of the coefficient matrix A(z) are single valued and analytic in the "star" region formed by non-intersecting straight lines from all the singularities and branch points of each of the $a_{ij}(z)$ to infinity. The point z_0 is within this star.

For z real we require instead that:

(b) The $a_{ij}(z)$ are single valued, bounded and integrable in an interval containing the point z_0.

Under either of the above conditions it has been proved by various methods that the solution of the one point boundary value problem stated by equation 1) exists and is unique. In the present paper we wish to exploit the method of Peano [1] and Baker [2,3,4] .

Following Baker [4] we define the square matrix of n-rows, which he has called the matrizant of A(z), by*

2) $$\Omega_{z_0}^{z}\left\{A(x)\right\} \equiv I + \int_{z_0}^{z} A(x_1)dx_1 + \int_{z_0}^{z} A(x_1) \int_{z_0}^{x_1} A(x_2)\, dx_2\, dx_1 + \ldots\ldots$$

---

* One is led to the same series by applying the Picard iteration scheme to the system (1). Or considering the equivalent system of integral equations

$$U(z) = U_0 + \int_{z_0}^{z} A(x)\,U(x)\,dx\ ,$$

the Neumann expansion of the solution yields the series (2). Also Volterra's [13] definition of the "left integral" of a matrix is precisely its matrizant.

Here I is the n x n unit matrix. In the complex case all integrals in (2) are over the same path which does not intersect any of the barriers of condition (a) but is otherwise arbitrary. The series for the elements of $\Omega_{z_0}^{z}\left\{A(x)\right\}$ converge uniformly and absolutely [5], and thus term by term differentiation is valid and yields

3) $$\frac{d}{dz}\,\Omega_{z_0}^{z}\left\{A(x)\right\} = A(z)\,\Omega_{z_0}^{z}\left\{A(x)\right\} .$$

Hence the solution of the system (1) is

4) $$U(z) = \Omega_{z_0}^{z}\left\{A(x)\right\} U_0 \quad .$$

This series converges uniformly and absolutely throughout the entire star region of (a) or in the interval of (b), which regions are much larger than those obtained by most other methods. This feature is compensated for by the fact that the convergence is usually so slow that the solution in this form is not practical for computation.

III. Transformation of the Solution

In order to transform the solution (4) to a more rapidly convergent form we will employ the following identities:

5) $$\Omega_{z_0}^{z}\left\{A(x)\right\} \equiv P(z)\,\Omega_{z_0}^{z}\left\{P^{-1}(x)\,A(x)\,P(x) - P^{-1}(x)\,\frac{dP(x)}{dx}\right\}P^{-1}(z_0)$$

6) $$\Omega_{z_0}^{z}\left\{A(x)\right\} \equiv \exp\int_{z_0}^{z} A(x)\,dx$$

7) $$\Omega_{z_0}^{z}\left\{X + Y\right\} \equiv \Omega_{z_0}^{z}\left\{X\right\}\,\Omega_{z_0}^{z}\left\{\Omega_{z_0}^{x-1}\left\{X\right\}\,Y(x)\,\Omega_{z_0}^{x}\left\{X\right\}\right\}$$

8) $$\Omega_{z_0}^{z-1}\left\{X(x)\right\} \equiv \Omega_{z}^{z_0}\left\{X(x)\right\}$$

9) $$\Omega_{z_0}^{z}\left\{A(x)\right\} = \Omega_{\xi}^{z}\left\{A(x)\right\}\,\Omega_{z_0}^{\xi}\left\{A(x)\right\}$$

Identity (5) holds for any non-singular matrix $P(z)$ which is analytic, in the complex case, and has a bounded integrable derivative, in the real case. Identity (6) holds for any matrix such that $A(z')\,A(z'') = A(z'')\,A(z')$ for all z' and z'' on the path of integration. In particular this applies to all diagonal and constant matrices. Identities (7), (8) and (9) hold for arbitrary matrices provided the matrizants exist. Identity (6) is proved in appendix I; the others appear in the literature [4,5,6].

Applying identity (5), the solution given by (4) may be written

10) $$U(z) = P(z)\,\Omega_{z_0}^{z}\left\{P^{-1}(x)\,A(x)\,P(x) - P^{-1}(x)\,\frac{dP(x)}{dx}\right\}P^{-1}(z_0)\,U_0 \,.$$

If $P(z)$ can be chosen such that $\left\{P^{-1}AP - P^{-1}\,\frac{dP}{dz}\right\}$ is diagonal, identity (6) leads to a solution in closed form. However, there is no general procedure for determining such a matrix P and therefore we merely attempt to diagonalize the first term, $P^{-1}\,AP$.

At any point z this is possible if and only if the elementary divisors of the matrix $(A(z) - \lambda I)$ are simple, where λ is a constant [7]. Then we have

$$11) \qquad P^{-1}(z)\, A(z)\, P(z) = \begin{pmatrix} \lambda_1(z) & & & 0 \\ & \lambda_2(z) & & \\ & & \ddots & \\ 0 & & & \lambda_n(z) \end{pmatrix} .$$

where the $\lambda_i(z)$ are the roots of the characteristic equation [7]

$$12) \qquad \det\,(A(z) - \lambda I) = 0 \quad .$$

We assume that there are regions in the star of condition (a) or the interval of condition (b) throughout which this diagonalization is possible and we restrict our attention to such regions.

We now introduce the abbreviation

$$13) \qquad R(z) \equiv -P^{-1}(z)\, \frac{dP(z)}{dz} \quad .$$

Then using equations (11) and (13) we may write

$$14) \quad \left\{ P^{-1}AP - P^{-1}\,\frac{dP}{dz} \right\} = \begin{pmatrix} \lambda_1 + r_{11} & & & 0 \\ & \lambda_2 + r_{22} & & \\ & & \ddots & \\ 0 & & & \lambda_n + r_{nn} \end{pmatrix} + \begin{pmatrix} 0 & r_{12} & \cdots & r_{1n} \\ r_{21} & 0 & & \vdots \\ \vdots & & \ddots & r_{n-1,n} \\ r_{n1} & \cdots & r_{n,n-1} & 0 \end{pmatrix}$$

Here the $r_{ij}(z)$ are the elements of $R(z)$. We use the equation (14) in the solution, equation (10), and expand the resulting matrizant with the aid of identities (6) and (7). This yields

$$15) \qquad U(z) = P(z) \cdot \exp \int_{z_0}^{z} L(x)dx \cdot \underset{z_0}{\overset{z}{\Omega}} \left\{ M(z_0, x) \right\} \cdot P^{-1}(z_0)\; U_0 \quad .$$

In this expression $L(x)$ is the diagonal matrix on the right side of equation (14) and has elements $\ell_j(x) \equiv \lambda_j + r_{jj}$. $M(z_0, z)$ is the matrix whose elements are

16) $$m_{ij}(z) \equiv (1-\delta_{ij})\ r_{ij}(z)\ e^{\int_{z_0}^{z}[\ell_j(x)-\ell_i(x)]\,dx} \quad .$$

The representation of the solution given by equation (15) is the desired rapidly convergent form which exhibits its behavior.

In order to express the solution more clearly we introduce the two column matrices $C(z_0,z)$ and $B(z_0)$ by the definitions

17) $$C(z_0,z) \equiv \Omega_{z_0}^{z}\{M(z_0,x)\}\ B(z_0) \qquad ; \qquad B(z_0) \equiv P^{-1}(z_0)\ U_0 \quad .$$

Further, the elements of $C(z_0,z)$ may be written as

18) $$C_i(z_0,z) = \sum_{s=1}^{n} \Omega_{is}(z_0,z)\ B_s(z_0) \qquad ; \qquad i = 1,2,\ \ldots,\ n \quad .$$

Here the $B_s(z_0)$ are the elements of the matrix $B(z_0)$ and the $\Omega_{ij}(z_0,z)$ are the elements of the matrizant in equation (17). These are, by the definitions (2) and (16)

19) $$\Omega_{ij}(z_0,z) \equiv \delta_{ij} + \int_{z_0}^{z} m_{ij}(x_1)dx_1 + \sum_{s=1}^{n}\int_{z_0}^{z} m_{is}(x_1)\int_{z_0}^{x_1} m_{sj}(x_2)dx_2dx_1 + \ldots\ldots$$

Using equations (17) in the solution (15) yields

20) $$U(z) = P(z)\cdot \exp\int_{z_0}^{z} L(x)\,dx \cdot C(z_0,z) \quad .$$

Thus a particular component of this solution may be written as

21) $$u_i(z) = \sum_{j=1}^{n}\left[p_{ij}(z)\ e^{\int_{z_0}^{z} r_{jj}(x)dx}\right] e^{\int_{z_0}^{z}\lambda_j(x)dx}\qquad C_j(z_0,z)\ ;\ i=1,2,\ \ldots,\ n.$$

Here $u_i(z)$ is a component of $U(z)$, the $p_{ij}(z)$ are the elements of $P(z)$, and the elements of $L(x)$ have been written explicitly.

Finally, by taking account of the definition of $r_{jj}(x)$ it is possible to write the solution given by equation (21) in the form

$$22) \qquad u_i(z) = \sum_{j=1}^{n} \left[\frac{p_{ij}(z) e^{\int_{z_o}^{z} q_{jj}(x)dx}}{\det P(z)} \right] e^{\int_{z_o}^{z} \lambda_j(x)dx} \quad C_j(z_o, z) \cdot \det P(z_o) \quad .$$

Here $q_{jj}(x)$ is defined by

$$23) \qquad q_{jj}(x) \equiv \frac{1}{\det P(z)} \sum_{i=1}^{n} p_{ij}(x) \frac{dQ_{ji}(x)}{dx} \qquad ,$$

where $Q_{ji}(x)$ is the cofactor of $p_{ij}(x)$ in det $P(x)$ (see appendix II).

Equation 22 yields the desired representation of the solution. Each component of the solution is a sum of exponential functions with variable exponents and variable coefficients. Each exponent is the integral of a latent root of the matrix A. Each coefficient is a product of a function in closed form and another, C_j, which is an infinite series of multiple integrals. If only the first term, a constant, in each C_j is retained the solution coincides with the WKB approximation in the second order case for which the latter is defined, (see section V). Thus the WKB approximation is seen to be the first term in a uniformly and absolutely convergent representation of the solution. Furthermore the leading terms in equation (22) can be considered a natural generalization of the WKB approximation to higher order systems. The successive terms in C_j can be simply interpreted as corresponding to multiple continuous reflection caused by the variation in A (see section V). This explains why a single exponential term (or a single wave) cannot yield a solution by itself: each wave produces other waves by continuous reflection. Only when the derivatives of A are zero (i.e. when A is constant) are these reflections absent, and thus in this case a single wave can constitute a solution.

IV. Connection Across Turning Points.

The representation of the solution given by equation (22) is not valid if there are points on the path of integration at which A cannot be diagonalized, and in the second order case these are just the turning points of the ordinary WKB method. Thus we will also call them turning points in the general case, and we will assume that they are isolated. In the complex case turning points offer no difficulty, because it is only necessary to restrict the path of integration to avoid them, as was done in the preceding section. Since any point which is not itself a turning point can be reached from the initial point z_0 by such a path, the representation given by equation (22) holds at all points except turning points. In the case of z real, however, the representation in equation (22) holds only when z and z_0 are not separated by a turning point. To extend the solution across a turning point, or to connect the representations on opposite sides of a turning point, it is necessary to employ a different representation of the solution. We will use the original matrizant representation for this purpose.

Thus, suppose there are turning points only in the interval $z_0 < z < z_1$, and we wish to obtain the solution at $\xi \geq z_1$. By the method of the preceding section this may be written as

$$24) \qquad U(\xi) = P(\xi) \exp \int_{z_1}^{\xi} L(y)dy \ \Omega_{z_1}^{\xi} \left\{ M(z_1, y) \right\} P^{-1}(z_1)\, U(z_1)$$

where all matrices are defined as above. However, this solution involves $U(z_1)$ and we are given only $U(z_0) = U_0$. Applying the solution in the form given by equation (4) we obtain

$$25) \qquad U(z_1) = \Omega_{z_0}^{z_1} \left\{ A(y) \right\} U_0 \quad .$$

Now the solution for $\xi \geq z_1$ may be written as:

$$26) \qquad U(\xi) = P(\xi) \exp \int_{z_1}^{\xi} L(y)dy \ D(z_1, \xi) \quad ;$$

where the column matrix $D(z_1, \xi)$ is defined by

$$27) \qquad D(z_1, \xi) \equiv \Omega_{z_1}^{\xi} \left\{ M(z_1, y) \right\} P^{-1}(z_1)\ \Omega_{z_0}^{z_1} \left\{ A(y) \right\} U_0 \quad .$$

As before a particular component of the solution (26) is

$$28) \qquad u_i(\xi) = \sum_{j=1}^{n} \left[\frac{p_{ij}(\xi) e^{\int_{z_1}^{z} q_{jj}(y)dy}}{\det P(\xi)} \right] e^{\int_{z_1}^{\xi} \lambda_j(y)dy} \quad D_j(z_1, \xi) \cdot \det P(z_1) \ ; \ i = 1, 2, \dots . n$$

Equations (27) and (28) give the solution for $\xi \geq z_1$ while equations (20)-(22) now apply only for $z \leq z_0$. The connection across the interval containing the turning points has been furnished by the matrizant appearing in equation (25). The solution for $\xi \geq z_1$ may be obtained from that for $z \leq z_0$ by determining the square matrix $T(\xi, z_1; z, z_0)$ such that

$$29) \qquad D(z_1, \xi) = T(\xi, z_1; z, z_0)\ C(z_0, z) \quad .$$

From equations (17) and (27) this matrix is found to be

$$30) \qquad T(\xi, z_1; z, z_0) \equiv \Omega_{z_1}^{\xi} \left\{ M(z_1, y) \right\} P^{-1}(z_1)\ \Omega_{z_0}^{z_1} \left\{ A(y) \right\} P(z_0)\ \Omega_{z}^{z_0} \left\{ M(z_0, y) \right\}$$

and this furnishes the connection between any points z and ξ in their respective ranges.

V. Scattering Matrix

The connection matrix T defined by equation (30) relates the amplitudes of the waves at any two points ξ and z, when the phases of the waves are referred to the points z_1 and z_o respectively. This matrix can, of course, be defined whether or not turning points are present; if there are none it becomes simpler since z_o and z_1 may then be identified. It is frequently useful to define another matrix related to T, the scattering matrix S. In order to define S it is first necessary to designate some of the waves at ξ and some at z as "incoming" or "incident" waves; the remainder are called "outgoing" or "scattered" waves. The scattering matrix yields the amplitudes of the scattered waves in terms of those of the incident waves. For this to be possible it is necessary that half of the 2n waves (n each at ξ and z) be incoming and the other half outgoing.

Let us suppose, then, that the roots are so ordered that the first k waves at z and the last n-k waves at ξ are incident waves. Thus we may write

$$C(z_o,z) = \begin{pmatrix} C^k \\ C^{n-k} \end{pmatrix} \equiv \begin{pmatrix} C_1 \\ \vdots \\ C_k \\ C_{k+1} \\ \vdots \\ C_n \end{pmatrix}$$

31)

$$D(z_1,\xi) = \begin{pmatrix} D^k \\ D^{n-k} \end{pmatrix} \equiv \begin{pmatrix} D_1 \\ \vdots \\ D_k \\ D_{k+1} \\ \vdots \\ D_n \end{pmatrix} .$$

Here C^k and D^{n-k} are the amplitudes of the incident waves; C^{n-k} and D^k are the amplitudes of the scattered waves. Equation (29) may now be written in block form as

$$\begin{pmatrix} D^k \\ D^{n-k} \end{pmatrix} = \begin{pmatrix} T_{kk} & T_{k,n-k} \\ T_{n-k,k} & T_{n-k,n-k} \end{pmatrix} \begin{pmatrix} C^k \\ C^{n-k} \end{pmatrix} . \qquad 32)$$

The scattering matrix S is defined by

$$33)\qquad \begin{pmatrix} D^{k} \\ \\ C^{n-k} \end{pmatrix} = S \begin{pmatrix} C^{k} \\ \\ D^{n-k} \end{pmatrix} .$$

From the last two equations we obtain

$$34)\qquad S(\xi, z_1; z, z_0) = \begin{pmatrix} \left\{ T_{k,k} - T_{k,n-k}\, T^{-1}_{n-k,n-k}\, T_{n-k,k} \right\} & \left\{ T_{k,n-k}\, T^{-1}_{n-k,n-k} \right\} \\ \left\{ -T^{-1}_{n-k,n-k}\, T_{n-k,k} \right\} & \left\{ T^{-1}_{n-k,n-k} \right\} \end{pmatrix}$$

provided that

$$35)\qquad \det T_{n-k,n-k} \neq 0 .$$

The scattering matrix S given by eq. (34) yields the amplitudes of the scattered waves at ξ and z in terms of the amplitudes of the incident waves at the same points, when the phases of the waves at ξ and z are referred to z_1 and z_o respectively. Its definition is independent of the presence or absence of turning points. In most cases of interest S approaches a limit as ξ and z approach plus and minus infinity respectively, and this limit is what is usually called the scattering matrix. It depends only upon the reference points z_1 and z_o.

VI. Asymptotic Expansion of the Solution.

If the matrix A(z) depends upon a parameter k in an appropriate way, then the solution of equations (1) possesses an asymptotic expansion in reciprocal powers of k in regions where the latent roots of A are distinct. This result was obtained by Birkhoff [8] for an nth order equation and has since been extended to nth order systems. The connection between the asymptotic expansions in different regions, when turning points are present, has been treated by Langer [9] for second order equations, but very little has been done on this problem for higher order systems. From the viewpoint of the present paper, the significance of Birkhoff's result is that the asymptotic expansion which he obtains is just a sum of exponential functions with variable coefficients and variable exponents. In the second order case the leading terms in the expansion coincide with the WKB approximation, thus proving that the WKB approximation yields the leading term in an asymptotic expansion of the solution if the equation depends upon a parameter appropriately. Similarly, in the higher order case the leading terms of our representation of the solution in eq. (22) coincide with the leading terms in an asymptotic expansion of Birkhoff's type.

Since the asymptotic expansion can be deduced easily from our form of solution, and since this derivation may clarify the role of the WKB approximation as the leading term in both convergent and asymptotic expansions, we will derive it. Suppose, then, that the matrix A depends upon a parameter k such that $A(z,k) \equiv k\,A(z)$ where A(z) is independent of k. The matrix P(z) is then chosen to diagonalize A(z) and hence it, $P^{-1}(z)$ and R(z) are also independent of k. The elements of the matrix $M(z_o,z)$ now become, from equation (16),

$$36)\qquad m_{ij}(k,z) \equiv (1-\delta_{ij})\left[R_{ij}(z)\,e^{\int_{z_o}^{z}(R_{jj}(z)-R_{ii}(z)dz}\right]e^{k\int_{z_o}^{z}(\lambda_j(z)-\lambda_i(z))dz} .$$

Here the $\lambda_i(z)$ are the eigenvalues of the matrix A(z). The solution of the system $\frac{dU(z)}{dz} = A(z,k)\,U(z)$ is from equation (21)

$$37)\qquad u_i(z,k) = \sum_{j=1}^{n}\left[P_{ij}(z)e^{\int_{z_o}^{z}R_{jj}(z)dz}\right]\cdot e^{k\int_{z_o}^{z}\lambda_j(z)dz}\cdot\left[\Omega_{j1}(z_o,z;k)B_1+\dots\right.$$

$$\left.\dots+\Omega_{jn}(z_o,z;k)B_n\right];\ i = 1,2,\ \dots,\ n .$$

The $\Omega_{ij}(z_o,z;k)$ are obtained by replacing the $m_{ij}(z)$ appearing in equation (19) by the $m_{ij}(k,z)$ of equation (36). The asymptotic expansion of this solution is determined by that of the $\Omega_{ij}(z_o,z;k)$.

Let us introduce the notation

38)

$$f_{ij}(z) \equiv (1-\delta_{ij}) \left[R_{ij}(z)\, e^{\int_{z_o}^{z}(R_{ij}(z)-R_{ii}(z))dz} \right]$$

$$g_{ij}(z) \equiv \int_{z_o}^{z} (\lambda_j(z)-\lambda_i(z))\, dz \quad ,$$

with $g_{ij}(z_o) = 0$ and $i \neq j$. Now we have

39)

$$\Omega_{ij}(z_o,z;k) \equiv \delta_{ij} + \int_{z_o}^{z} f_{ij}(x_1)\, e^{kg_{ij}(x_1)}\, dx_1$$

$$+ \sum_{r=1}^{n} \int_{z_o}^{z} f_{ir}(x_1)\, e^{kg_{ir}(x_1)} \int_{z_o}^{x_o} f_{rj}(x_2)\, e^{kg_{rj}(x_2)} dx_2\, dx_1 + \ldots\ldots$$

We wish to integrate by parts (m+1) times in each of the above terms; this requires that the $f_{ij}(z)$ and $g_{ij}(z)$ be differentiable (m+1) and (m+2) times respectively. In case (a) this condition is automatically satisfied but case (b) must be amended to include differentiability of order (m+2) of the $a_{ij}(z)$. To prove the asymptotic nature of the resulting expansion we must restrict ourselves to that portion of the complex plane or real interval where

40) $$\mathrm{Re}\left[kg_{ij}(z)\right] \leq 0$$

for all $g_{ij}(z)$. But from the definition (38) it is seen that $g_{ij}(z) = -\, g_{ji}(z)$ and so condition (40) implies

$$\mathrm{Re}\left[kg_{ij}(z)\right] = 0 \quad .$$

Under the above conditions we may now integrate by parts (m+1) times in the first integral of equation (39). This yields

$$\int_{z_o}^{z} f_{ij}(y)e^{kg_{ij}(y)}dy = \frac{e^{kg_{ij}(z)}}{k}\sum_{r=0}^{m}\frac{\alpha_r^{(i,j)}(z)}{k^r} - \frac{1}{k}\sum_{r=0}^{m}\frac{\alpha_r^{(ij)}(z_o)}{k^r}$$

41)
$$+\ \frac{1}{k^{m+1}}\int_{z_o}^{z}\frac{d\,\alpha_m^{(i,j)}(y)}{dy}\ e^{kg_{ij}(y)}\,dy\ ;$$

where

$$\alpha_o^{(i,j)}(z) = \left(\frac{dg_{ij}(z)}{dz}\right)^{-1} f_{ij}(z)\ ;\ \alpha_r^{(i,j)}(z) = -\left(\frac{dg_{ij}(z)}{dz}\right)^{-1}\frac{d\,\alpha_{r-1}^{(i,j)}(z)}{dz},\ r=1,2,\dots,m.$$

Here we have used the relation $\frac{dg_{ij}(z)}{dz} = \lambda_j(z)-\lambda_i(z) \neq 0$, which is obvious from (38) and the absence of turning points. By condition (40) it is now apparent that for $|k| \to \infty$ the first integral of equation (39) has the asymptotic expansion given by the first 2m terms on the right of equation (41). Similarly it is easily demonstrated that the p-th iterated integral terms in (39) have asymptotic expansions in powers of $\frac{1}{k}$ starting with terms of $\frac{1}{k^p}$. By adding all of these expansions we finally obtain the asymptotic expansion of $\Omega_{ij}(z_o,z;k)$ and hence of the solution (37).

The terms multiplying the second square bracket of equation (37) constitute what we have called in the general case, or is exactly for the second order equation, the W.B.K. solution. Further these square brackets have asymptotic expansions which consist of a constant plus a power series in $\frac{1}{k}$. Thus it is shown that in general the W.B.K. solution is the leading term in the asymptotic expansion of the rigorous solution.

VII. Second Order Equation.

We shall now apply the preceeding analysis to the second order differential equation

42) $$\frac{d^2\psi(z)}{dz^2} - 2\,a(z)\,\frac{d\psi(z)}{dz} + b(z)\psi(z) = 0$$

with the one point boundary conditions

43) $$\psi(z_o) = \psi^o \quad ; \quad \left.\frac{d\psi(z)}{dz}\right|_{z=z_o} = \psi^1 \quad .$$

This equation with the boundary conditions may be put into the form of the system (1) by introducing $u_1(z) = \psi(z)$; $u_2(z) = \frac{d\psi(z)}{dz}$ and the matrices:

44) $$A(z) \equiv \begin{pmatrix} 0 & 1 \\ -b(z) & 2a(z) \end{pmatrix}, \quad U(z) = \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix}, \quad U_o = \begin{pmatrix} \psi^o \\ \psi^1 \end{pmatrix}$$

To solve the resulting equation by the method of sections II and III we are led to consider the algebraic equation

$$\left| \lambda I - A(z) \right| \equiv \lambda^2 - 2a(z)\,\lambda + b(z) = 0 \quad .$$

This has the two solutions

45) $$\left.\begin{aligned} \lambda_1(z) &= a(z) - i\mu(z) \\ \lambda_2(z) &= a(z) + i\mu(z) \end{aligned}\right\} \mu(z) \equiv \sqrt{b(z) - a^2(z)}$$

The conditions (a) and (b) of section II are extended to exclude all values of z such that $b(z) = a^2(z)$. The matrix which diagonalizes $A(z)$ is found to be

46) $$P(z) \equiv \lambda_2(z) \begin{pmatrix} -1 & \frac{1}{\lambda_2(z)} \\ -\lambda_1(z) & 1 \end{pmatrix}, \quad \text{with } P^{-1}(z) \equiv \frac{1}{\lambda_1(z) - \lambda_2(z)} \begin{pmatrix} 1 & -\frac{1}{\lambda_2(z)} \\ \lambda_1(z) & -1 \end{pmatrix};$$

and thus

$$47)\quad R(z) \equiv \begin{pmatrix} -\left[\dfrac{\dot\lambda_2(z)}{\lambda_2(z)} + \dfrac{\dot\lambda_1(z)}{(\lambda_1(z)-\lambda_2(z))}\right] & \left[\dfrac{\dot\lambda_2(z)}{\lambda_2(z)(\lambda_1(z)-\lambda_2(z))}\right] \\ -\left[\dfrac{\lambda_2(z)\dot\lambda_1(z)}{(\lambda_1(z)-\lambda_2(z))}\right] & \left[\dfrac{\dot\lambda_2(z)}{(\lambda_1(z)-\lambda_2(z))}\right] \end{pmatrix} .$$

In equation (47) and the following the dots denote differentiation. The solution developed in section III becomes

$$48)\quad u_1(z) = e^{\int_{z_0}^{z} a(y)dy} \left\{ \frac{e^{-i\int_{z_0}^{z}\mu(y)dy}}{\sqrt{\mu(z)}}\, e^{-\frac{i}{2}\int_{z_0}^{z}\frac{\dot a(y)}{\mu(y)}dy} \left[\Omega_{11}(z)B_1 + \Omega_{12}(z)B_2\right] (-\lambda_2(z_0)) \right.$$

$$\left. + \frac{e^{i\int_{z_0}^{z}\mu(y)dy}}{\sqrt{\mu(z)}}\, e^{\frac{i}{2}\int_{z_0}^{z}\frac{\dot a(y)}{\mu(y)}dy} \left[\Omega_{21}(z)B_1 + \Omega_{22}(z)\, B_2\right] \right\} \sqrt{\mu(z_0)}$$

with a similar expression for $u_2(z)$. Here the $\Omega_{ij}(z)$ are expressed by equation (19) where

$$m_{11}(z) = m_{22}(z) = 0$$

$$m_{21}(z) = \lambda_2^2(z_0)\, m_{12}^*(z)$$

$$49)\quad m_{12}(z) = \frac{1}{\lambda_2(z_0)}\,\frac{1}{2}\left[-\frac{\dot\mu(z)}{\mu(z)} + i\,\frac{\dot a(z)}{\mu(z)}\right] e^{i2\int_{z_0}^{z}\left[\mu(y) + \frac{1}{2}\frac{\dot a(y)}{\mu(y)}\right]dy}$$

and $m_{12}^*(z)$ is the complex conjugate of $m_{12}(z)$.

If in equation (42) we take $a(z) \equiv 0$ the resulting equation, $\ddot\psi(z) + b(z)\,\psi(z) = 0$, has many important physical applications. In particular it represents the reduction of Maxwell's equations for plane electromagnetic waves in an isotropic plane stratified inhomogeneous medium. The solution now becomes

$$\psi(z) = \left\{ \frac{e^{-i\int_{z_0}^{z}\mu(y)dy}}{\sqrt{\mu(z)}} \left[\Omega_{11}(z)B_1 + \Omega_{12}(z)\,B_2\right]\left(-\mu(z_0)\right) \right.$$

50)

$$\left. + \frac{e^{i\int_{z_0}^{z}\mu(y)dy}}{\sqrt{\mu(z)}} \left[\Omega_{21}(z)B_1 + \Omega_{22}(z)\,B_2\right] \right\} \sqrt{\mu(z_0)}$$

where the $\Omega_{ij}(z)$ are defined as before, with

$$m_{11}(z) = m_{22}(z) = 0$$

51)

$$m_{12}(z) = \frac{-1}{\mu(z_0)}\left[\frac{d}{dz}\ln\mu(z)^{1/2}\right] e^{i2\int_{z_0}^{z}\mu(y)dy}$$

$$m_{21}(z) = -\mu(z_0)\left[\frac{d}{dz}\ln\mu(z)^{1/2}\right] e^{-i2\int_{z_0}^{z}\mu(y)dy} \quad ;$$

and $\mu(z) = \sqrt{b(z)}$.

If the integrals in the $\Omega_{ij}(z)$ are small then it is clearly shown that the leading term of the above solution is the ordinary W.B.K. solution. Also the usual turning points occur when $\mu(z) = 0$ which correspond to equal roots in the present treatment. Considering the electromagnetic problem mentioned at the beginning of this section, the physical interpretation becomes quite clear. The positive and negative exponentials represent "rising" and "downcoming" waves respectively. A particular $\Omega_{ij}(z)$, say $\Omega_{11}(z)$, is explicitely

$$\Omega_{11}(z)=1+\int_{z_0}^{z}\left[\frac{d}{dx_1}\ln\mu(x_1)^{\frac{1}{2}}\right]e^{i2\int_{z_0}^{z}\mu(y)dy}\int_{z_0}^{x_1}\left[\frac{d}{dx_2}\ln\,(x_2)^{\frac{1}{2}}\right]e^{-i2\int_{z_0}^{x_2}\mu(y)dy}\,dx_2dx_1+\ldots$$

This series represents all contributions of even numbers of continuous multiple reflections to the upgoing wave. The term $\Omega_{12}(z)$ yields the contributions to the upgoing wave of all such odd numbers of continuous reflections. Similarly for the remaining $\Omega_{ij}(z)$. Thus we can define a continuous reflection coefficient $\rho(z)$ by:

$$\rho(z) = \frac{d}{dz} \ln \mu(z)^{\frac{1}{2}} \quad . \tag{52}$$

The solution of the present section is very similar to that obtained by Bremmer for the electromagnetic problem [10].

VIII. Calculation of Connection Matrix.

The solution of the previous section fails when $\lambda_1(z) = \lambda_2(z)$. In the case of $a(z) \equiv 0$, this occurs for $b(z) = 0$. Let this turning point be located at $z = 0$ and in the neighborhood of this point we take

53) $$b(z) = k^2 z^\alpha \quad , \alpha > 0 \ .$$

Then the A matrix of the equivalent first order system is

54) $$A(z) \equiv \begin{pmatrix} 0 & 1 \\ -k^2 z^\alpha & 0 \end{pmatrix} .$$

The solution for $z > 0$ and $\xi < 0$ is given by equation (50) with the appropriate B_1 in each case. These solutions are connected, as explained in section IV, by the matrizant

55) $$\begin{aligned} t(\xi, z) \equiv \Omega_{\xi}^{z}\left\{A(x)\right\} &= \Omega_{0}^{z}\left\{A(x)\right\} \Omega_{\xi}^{0}\left\{A(x)\right\} \\ &= \Omega_{0}^{z}\left\{A(x)\right\} \Omega_{0}^{\xi\,-1}\left\{A(x)\right\} . \end{aligned} \quad .$$

Here we have used the identities (8) and (9). Thus we need compute only $\Omega_{0}^{z}\left\{A(x)\right\}$, whose elements are, from equations (53) and (2)

$$\Omega_{11}(z) = 1 - k^2 \int_0^z \int_0^{x_2} x_1^\alpha dx_1 dx_2 + k^4 \int_0^z \int_0^{x_4} \int_0^{x_3} \int_0^{x_2} x_3^\alpha x_1^\alpha dx_1 dx_2 dx_3 dx_4 - + \ldots\ldots$$

$$\Omega_{22}(z) = 1 - k^2 \int_0^z \int_0^{x_2} x_2^\alpha dx_1 dx_2 + k^4 \int_0^z \int_0^{x_4} \int_0^{x_3} \int_0^{x_2} x_4^\alpha x_2^\alpha dx_1 dx_2 dx_3 dx_4 - + \ldots\ldots$$

56) $$\Omega_{12}(z) = \frac{-z^{-\alpha}}{k^2} \frac{d}{dz} \Omega_{22}(z)$$

$$\Omega_{21}(z) = \frac{d}{dz} \Omega_{11}(z) \qquad .$$

The first two of the above series yield upon integration

$$\Omega_{11}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\left[k^2 z^{(\alpha+2)}\right]^n}{n!(\alpha+2)^n \prod_{m=1}^{n}\left[m(\alpha+2)-1\right]}$$

57)

$$\Omega_{22}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\left[k^2 z^{(\alpha+2)}\right]^n}{n!(\alpha+2)^n \prod_{m=1}^{n-1}\left[m(\alpha+2)+1\right]}$$

Applying the identities [11]

$$\prod_{m=1}^{n}\left[ma-1\right] \equiv \frac{a^n \Gamma\left(\frac{a-1}{a}+n\right)}{\Gamma\left(\frac{a-1}{a}\right)}$$

$$\prod_{m=1}^{n-1}\left[ma+1\right] \equiv \frac{a^{n-1} \Gamma\left(\frac{1}{a}+n\right)}{\Gamma\left(\frac{a+1}{a}\right)}$$

to equations (57) we obtain [11]

$$\Omega_{11}(z) = \Gamma(1-\nu)(\nu k)^{\nu}\, z^{\frac{1}{2}} J_{-\nu}\left(2\nu k z^{\frac{1}{2\nu}}\right)$$

58) $\quad ; \quad \nu = \frac{1}{\alpha+2}$.

$$\Omega_{22}(z) = \Gamma(\nu)(\nu k)^{1-\nu}\, z^{\frac{1}{2}\left(\frac{1-\nu}{\nu}\right)} J_{\nu-1}\left(2\nu k z^{\frac{1}{2\nu}}\right)$$

From the last two of equations (56) and equations (58) there results

$$\Omega_{21}(z) = -k\,\Gamma(1-\nu)(\nu k)^{\nu}\, z^{\frac{1}{2}\left(\frac{1-\nu}{\nu}\right)} J_{1-\nu}\left(2\nu k z^{\frac{1}{2\nu}}\right)$$

59)

$$\Omega_{12}(z) = \frac{1}{k}\,\Gamma(\nu)(\nu k)^{1-\nu}\, z^{\frac{1}{2}} J_{\nu}\left(2\nu k z^{\frac{1}{2\nu}}\right) \quad .$$

The elements of $\overset{\xi\,-1}{\underset{0}{\Omega}}\{A\}$ are easily computed by finding the inverse of $\overset{z}{\underset{0}{\Omega}}\{A\}$ and replacing z by ξ. Then we obtain the connection matrix of equation (55) as

60)
$$t(\xi,z)=\begin{pmatrix} z^{\frac{1}{2}}\xi^{\frac{\alpha+1}{2}}\left[J_{-\nu}(z')J_{\nu-1}(\xi')-J_{\nu}(z')J_{1-\nu}(\xi')\right] & (z\xi)^{\frac{1}{2}}\left[J_{\nu}(z')J_{-\nu}(\xi')-J_{-\nu}(z')J_{\nu}(\xi')\right]\frac{1}{k} \\ (z\xi)^{\frac{m+1}{2}}\left[J_{1-\nu}(z')J_{\nu-1}(\xi')-J_{\nu-1}(z')J_{1-\nu}(\xi')\right] & z^{\frac{m+1}{2}}\xi^{\frac{1}{2}}\left[J_{\nu-1}(z')J_{-\nu}(\xi')-J_{1-\nu}(z')J_{\nu}(\xi')\right] \end{pmatrix}.$$

Here we have neglected the constant factor $\Gamma(1-\nu)\Gamma(\nu)(\nu k)$ and for the arguments of the Bessel functions the abbreviated notation

61)
$$z' = 2\nu k z^{\frac{1}{2\nu}}$$
$$\xi' = 2\nu k \xi^{\frac{1}{2\nu}}$$

has been used. This matrix furnishes the exact connection between rigorous solutions on either side of a turning point (of the form given in equation (53)). What are usually referred to as connection formulae are concerned with connecting asymptotic solutions across a turning point. These may be obtained from the matrix (60) by using the asymptotic forms of the Bessel functions for large k and taking account of the Stokes phenomenom which arises since z' and ξ' have different arguments [12].

Appendix I. Matrizant of a "self-Commuting" Matrix.

To prove identity (6) we consider a "self-commuting" matrix $A(x)$; i.e. a matrix such that

$$A(x') \, A(x'') = A(x'') \, A(x') \tag{62}$$

for arbitrary x' and x'' in some interval. It is assumed that the matrizant of $A(x)$ exists. From the definition (2) this may be written as

$$\overset{x}{\underset{x_0}{\Omega}} \left\{ A(x) \right\} \equiv I + \sum_{n=1}^{\infty} \int_{x_0}^{x} A(x_n) \int_{x_0}^{x_n} A(x_{n-1}) \int_{x_0}^{x_{n-1}} \cdots\cdots \int_{x_0}^{x_2} A(x_1) dx_1 dx_2 \cdots dx_n \; . \tag{63}$$

Now consider:

$$\begin{aligned} \frac{d}{dx} \left[\int_{x_0}^{x} A(\xi) d\xi \right]^{p} &= \sum_{s=1}^{p} \left[\int_{x_0}^{x} A(\xi) d\xi \right]^{p-s} A(x) \cdot \left[\int_{x_0}^{x} A(\xi) d\xi \right]^{s-1} \\ &= p \left[\int_{x_0}^{x} A(\xi) d\xi \right]^{p-1} A(x) \\ &= p \; A(x) \left[\int_{x_0}^{x} A(\xi) d\xi \right]^{p-1} \; . \end{aligned} \tag{64}$$

The first line of (64) follows from the ordinary formula for the deviative of a power of a matrix. The last two results are obtained by applying the property (62). The general term in the sum (63) may now be evaluated with the aide of the final result in (64). This yields

$$\begin{aligned} \overset{x}{\underset{x_0}{\Omega}} \left\{ A(x) \right\} &= I + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_{x_0}^{x} A(\xi) d\xi \right]^{n} \\ &= \exp \int_{x_0}^{x} A(\xi) d\xi \; . \end{aligned} \tag{65}$$

Appendix II. Non-Simple Elementary Divisors.

The case of multiple latent roots may be treated quite generally as follows. Let there be r roots each of multiplicity s_i, $i=1,2,\dots,r$; so that $\sum_{i=1}^{r} s_i = n$. Or in other words, the r-elementary divisors of the matrix $(A-\lambda I)$ have orders $s_1, s_2, \dots, s_r$ respectively. In this case it is known that there exists a non-singular matrix P(z) such that

$$66)\qquad P^{-1}(z)\,A(z)\,P(z) = B(z) \equiv \begin{pmatrix} B_1 & & 0 \\ & B_2 \ddots & \\ 0 & & B_r \end{pmatrix} .$$

Here each $B_i(z)$ is of order s_i with

$$67)\qquad B_i(z) \equiv \begin{pmatrix} \lambda_i(z) & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i(z) & 1 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & & & & \\ 0 & \cdots & \cdots & 0 & \lambda_i(z) \end{pmatrix}$$

and $\lambda_i(z)$ is a root of the characteristic equation (not all $\lambda_i(z)$ need be distinct).

We use the above matrix P(z) in the solution given by equation (10) and then apply identity (7) to obtain

$$68)\qquad U(z) = P(z)\,\Omega_{z_0}^{z}\left\{ B+D \right\}\,\Omega_{z_0}^{z}\left\{ M(x) \right\}\,P^{-1}(z_0)U_0$$

Again we have introduced:

$$69)\qquad P^{-1}(z)\,\frac{dP(z)}{dz} \equiv R(z) \equiv D(z) + N(z) \quad , \quad \text{and}$$

$$M(z) \equiv \Omega_{z}^{z_0}\left\{ B+D \right\} N(z)\,\Omega_{z_0}^{z}\left\{ B+D \right\}$$

where D is diagonal and N has zero diagonal elements. From the definition of the matrizant it is easily proven that:

70)
$$\underset{z_0}{\overset{z}{\Omega}}\left\{B+D\right\} = \begin{pmatrix} \underset{z_0}{\overset{z}{\Omega}}\left\{B_1+D_1\right\} & & & 0 \\ & \underset{z_0}{\overset{z}{\Omega}}\left\{B_2+D_2\right\} & & \\ & & \ddots & \\ 0 & & & \underset{z_0}{\overset{z}{\Omega}}\left\{B_r+D_r\right\} \end{pmatrix}$$

Here the $D_i(z)$ are (diagonal) sub-matrices of D, each of the same order as the associated $B_i(z)$. The identity (7) applied to each of the matrizants on the right of equation (70) gives

71)
$$\underset{z_0}{\overset{z}{\Omega}}\left\{B_i+D_i\right\} = \underset{z_0}{\overset{z}{\Omega}}\left\{B_i\right\} \underset{z_0}{\overset{z}{\Omega}}\left\{ \underset{z}{\overset{z_0}{\Omega}}\left\{B_i\right\} D_i \underset{z_0}{\overset{z}{\Omega}}\left\{B_i\right\}\right\}.$$

The matrizant of $B_i(z)$, which is given by equation (67), can be calculated from the definition; this yields

72)
$$\underset{z_0}{\overset{z}{\Omega}}\left\{B_i(x)\right\} = e^{\int_{z_0}^{z}\lambda_i(x)dx} \begin{pmatrix} 1 & (z-z_0) & \frac{1}{2}(z-z_0)^2 & \cdots\cdots & \frac{1}{s_i-1}(z-z_0)^{s_i-1} \\ & 1 & (z-z_0) & \ddots & \vdots \\ & & 1 & \ddots & \vdots \\ 0 & & & 1 & (z-z_0) \\ & & & & 1 \end{pmatrix}$$

Now equations (70)-(72) can be used in the solution (68), but the resulting form of the solution will contain essentially two matrizants which must be evaluated. It will be recalled that in the case of complete diagonalization (simple elementary divisors) the final form of the solution involved the evaluation of only one matrizant.

Appendix III. Elements of $P^{-1}\ \frac{dP}{dz}$

Let the elements of a non-singular matrix $P(z)$ be $p_{ij}(z)$. Then the elements $P_{ij}(z)$ of the inverse matrix $P^{-1}(z)$ are

$$P_{ij}(z) = \frac{1}{|P(z)|}\ Q_{ij}(z) \tag{73}$$

where

$$Q_{ij}(z) \equiv \left|p_{rs}(z)\right| \quad , \begin{cases} r \neq j \\ s \neq i \end{cases} \tag{74}$$

$$\equiv \text{ cofactor of } p_{ji}(z) \text{ in } |P(z)| .$$

The elements $R_{ij}(z)$ of the matrix $(-P^{-1}\ \frac{dP}{dz})$ are thus

$$R_{ij}(z) \equiv -\sum_{s=1}^{n} \frac{dp_{sj}(z)}{dz}\ P_{is}(z) \tag{75}$$

$$\equiv \frac{-1}{|P(z)|}\sum_{s=1}^{n} \frac{dp_{sj}(z)}{dz}\ Q_{is}(z) \quad .$$

However, from the definition (73):

$$\sum_{s=1}^{n} p_{sj}(z)\ Q_{is}(z) = \delta_{ij}\ |P(z)| \tag{76}$$

thus:

$$\sum_{s=1}^{n} \frac{dp_{sj}}{dz}\ Q_{is} = \delta_{ij}\ \frac{d}{dz}|P(z)| - \sum_{s=1}^{n} p_{sj}\ \frac{dQ_{is}}{dz} \quad . \tag{77}$$

Using (77) in (75) we obtain

$$R_{ij}(z) = -\ \delta_{ij}\ \frac{d}{dz}\ln|P(z)| + \frac{1}{|P(z)|}\sum_{s=1}^{n} p_{sj}\ \frac{dQ_{is}}{dz} \quad . \tag{78}$$

Letting $i = j$ the result of equations (22) and (23) follows

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